Scale-by-scale energy budgets in turbulent convection

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Talk outline

- Motivations
  - Turbulence in the lab...and in nature
  - Solar convection
- Scaling laws and thermal convection
  - Theories and experimental results
- This work
  - Generalized Kolmogorov and Yaglom equations
    - Forcing, inhomogeneity and anisotropy
  - Numerical study: Bolgiano or not Bolgiano ?
- Comparison with other results
  - Homogeneous shear flow & Channel flow

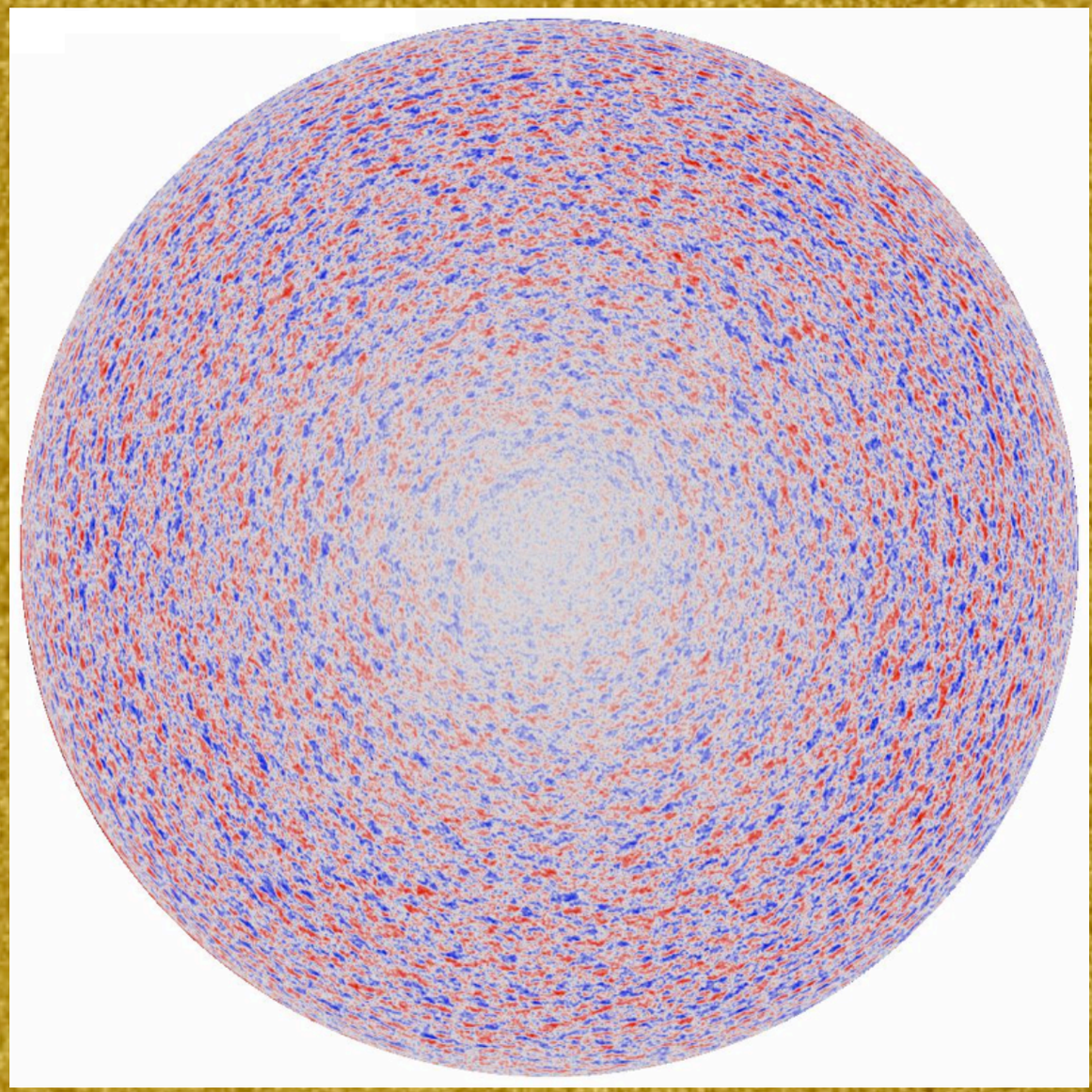
Studiis Rerum Naturae

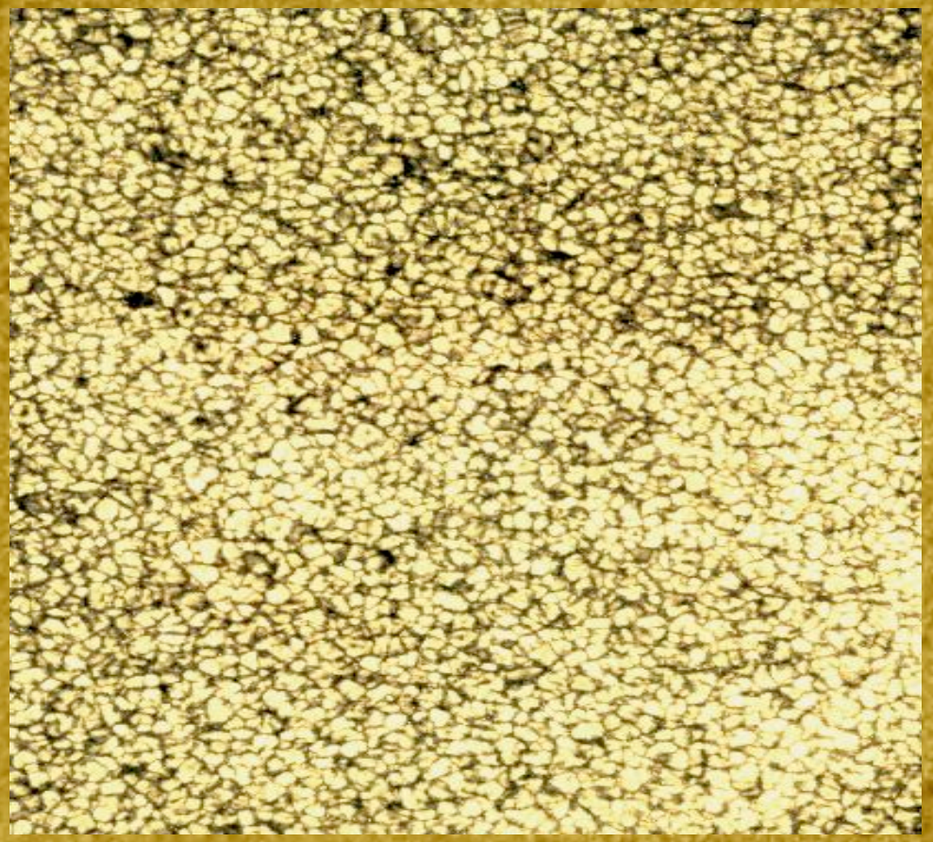
- Turbulence in nature is apparently diverse
  - Clouds, boundary layers, rivers, flow past obstacle, solar CZ, earth interior...
  - Variety of forcing processes & geometries: universality ?
- Turbulence in nature is often asymptotic
  - In the convective Sun: Re ~10^{10}, Ra~10^{20}, Pr~10^{-8}, Pm~10^{-5}
- Energy spectra are sometimes observable
  - Velocity, temperature spectra in solar convection
- Outstanding turbulence laboratories !
  - Scaling-law ranges may tell a lot about the physics of these objects
  - Large-scale forced, homogeneous, isotropic turbulence ?

Convection in the solar photosphere

- Turbulent convection observed “from above”
- Multiple horizontal scales
  - granulation (1Mm)
  - mesogranulation (5-8 Mm) ?
  - supergranulation (20-30 Mm)

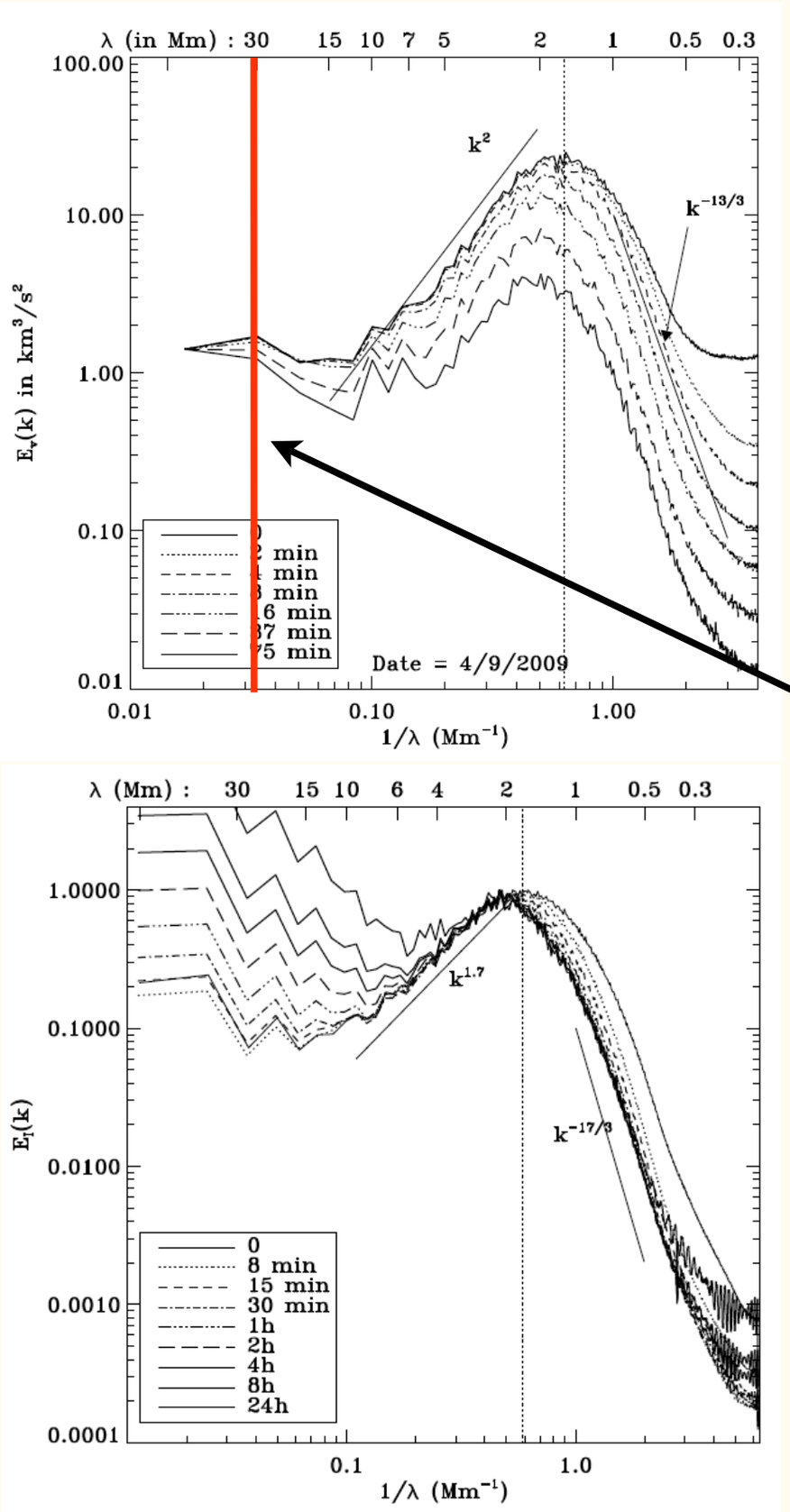
(MDI/SOHO)





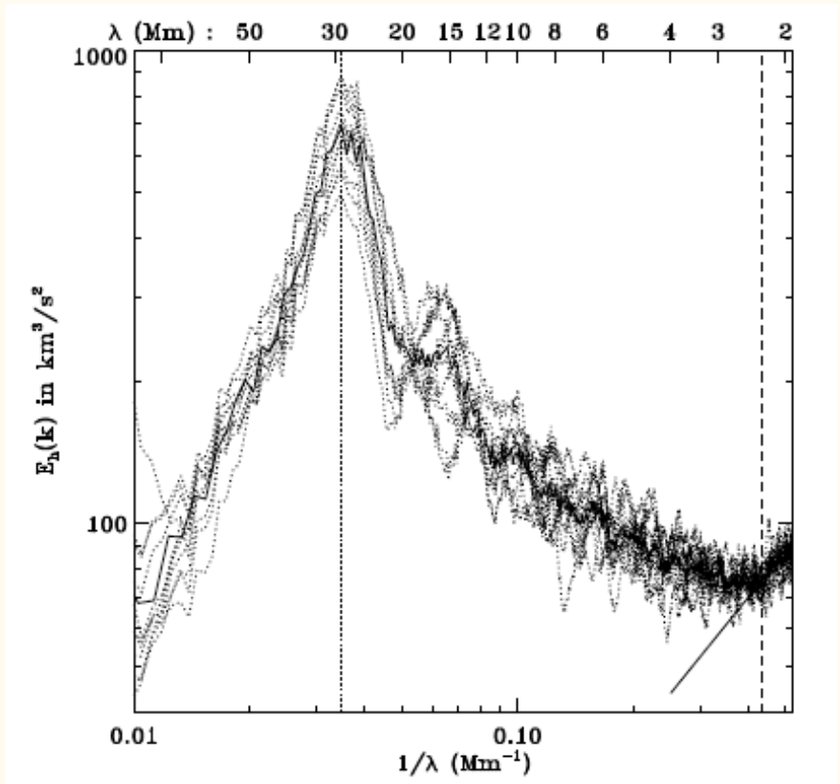
(LJR, CALAS, Pic du Midi)

Power spectra of solar surface flows



Vertical velocity:

$$E_v(k_h) \sim k_h^{-13/3}$$



Horizontal velocity
the supergranulation peak

Intensity:

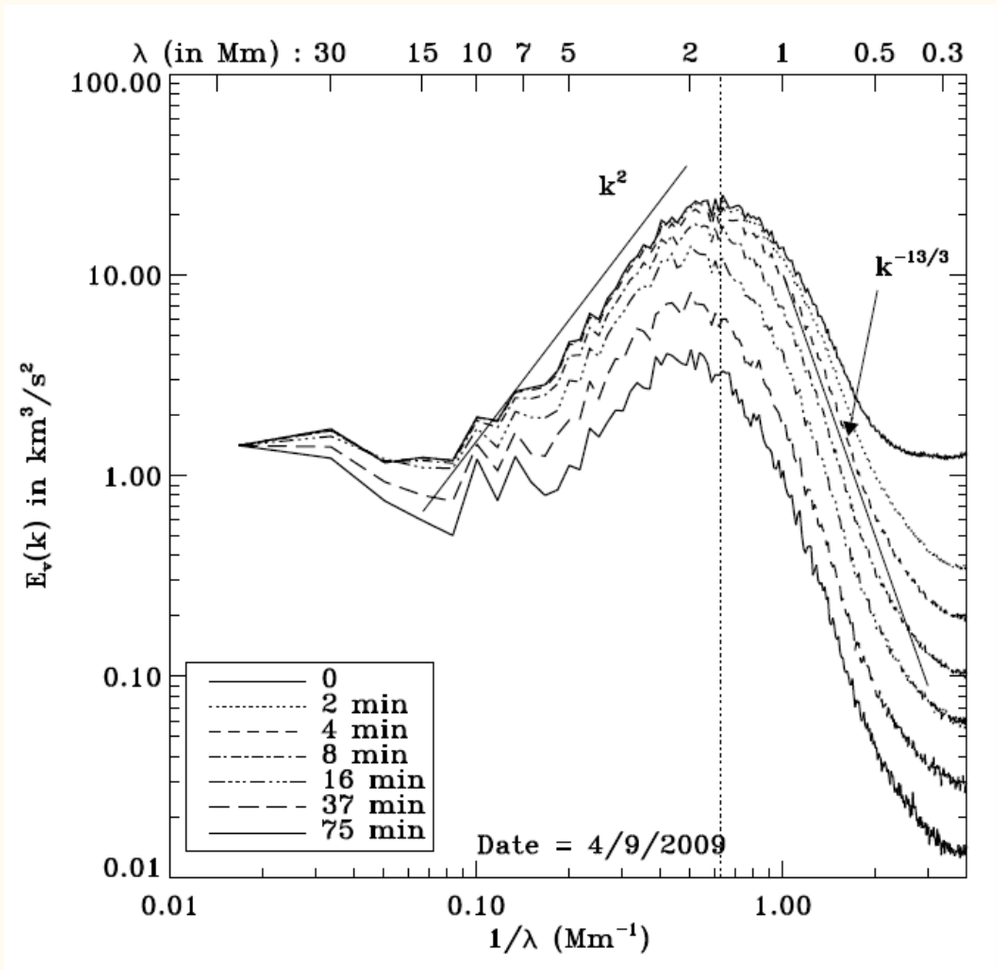
$$E_\theta(k_h) \sim k_h^{-17/3}$$

Rieutord, Roudier, Rincon et al.,
A&A 2009 accepted

Challenges in solar turbulent convection

- Turbulence models needed for solar physics
  - turbulent heat flux, coupling with oscillations, etc.
- Re >>1 : agreement with turbulence theories ?
  - gravity → forcing
  - gravity → anisotropy
    - plumes / cells / eddies
  - τ=1 boundary → inhomogeneity

Expected spectral scaling laws ?



Theories of turbulent convection

- Kolmogorov (K41): "4/5 law"

$$\left\langle (\Delta u_r)^3 \right\rangle = -\frac{4}{5} \left\langle \varepsilon \right\rangle r + 6\nu \frac{\partial}{\partial r} \left\langle (\Delta u_r)^2 \right\rangle$$

$$\left\langle (\Delta u_r)^p \right\rangle \sim r^{p/3} \qquad \left\langle (\Delta \theta)^p \right\rangle \sim r^{p/3}$$

- "Generalize" Kolmogorov and Yaglom equations (Yakhot, PRL 1992)

$$\left\langle (\Delta u_r)^3 \right\rangle = -\frac{4}{5} \left\langle \varepsilon \right\rangle r + \frac{6\alpha g}{r^4} \int_0^r y^4 \left\langle \Delta\theta \Delta u_z \right\rangle \mathrm{d}y + 6\nu \frac{\partial}{\partial r} \left\langle (\Delta u_r)^2 \right\rangle$$

$$\left\langle (\Delta\theta)^2 \Delta u_r \right\rangle = -\frac{4}{3} \left\langle N \right\rangle r + \frac{2}{r^2} \int_0^r y^2 \left\langle \Delta u_z \Delta\theta \right\rangle \mathrm{d}y \frac{\partial \left\langle T \right\rangle}{\partial z} + 2\kappa \frac{\partial}{\partial r} \left\langle (\Delta\theta)^2 \right\rangle$$

- Bolgiano-Obhoukov (BO59) theory

$$\left\langle (\Delta u_r)^p \right\rangle \sim r^{3p/5} \qquad \left\langle (\Delta \theta)^p \right\rangle \sim r^{p/5}$$

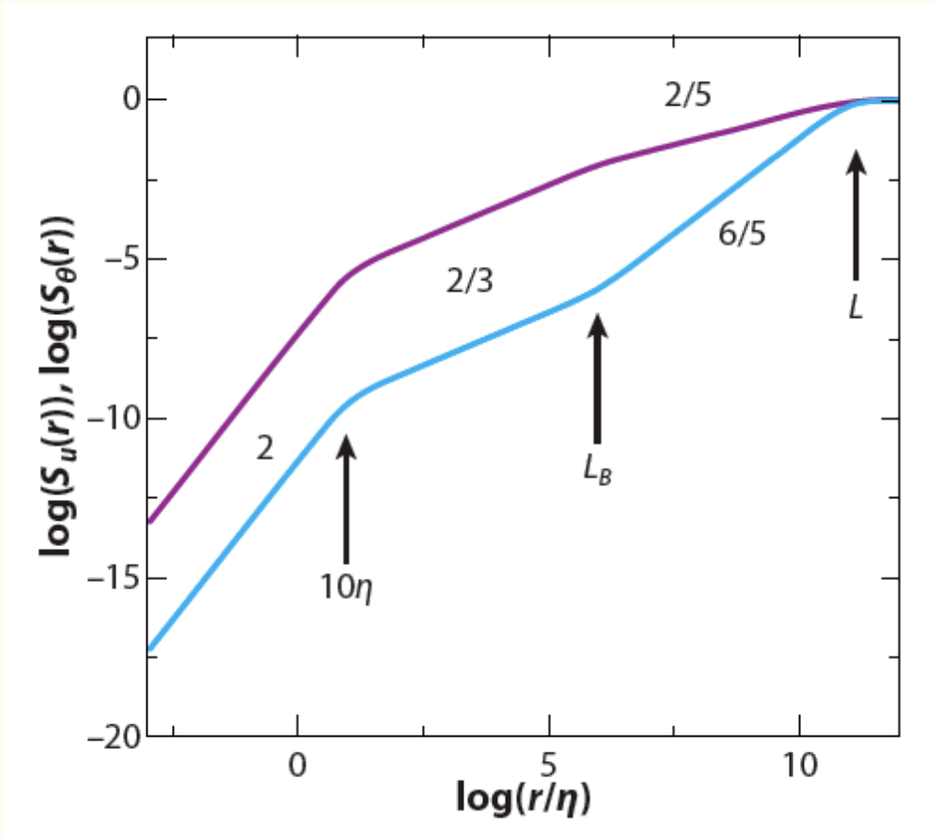
Theories of turbulent convection

- Corresponding spectral scalings
  - K41: $E(k) \sim k^{-5/3}$ $E_\theta(k) \sim k^{-5/3}$
  - BO59: $E(k) \sim k^{-11/5}$ $E_\theta(k) \sim k^{-7/5}$

- Bolgiano length (Chilla et al., NC 1993):

$$L_B = \frac{\langle \varepsilon \rangle^{5/4}}{\langle N \rangle^{3/4} (\alpha g)^{3/2}} \sim \frac{\mathrm{Nu}^{1/2} d}{(\mathrm{RaPr})^{1/4}}$$

  - BO59 if $r > L_B$
  - K41 if $\eta < r < L_B$



Lohse & Xia, ARFM 2010 in press

Experimental & numerical results

- The existence of a BO59 range is in debate
- Experiments
  - Yes: Chillá et al. (NC, 1993), Niemela et al. (Nat, 2000) // No: Sun et al. (PRL, 2006)
  - Pr-dependent: Cioni et al. (EL, 1995)
  - Cross-scalings between T and V: Ching et al. (J. Turb, 2004), Zhou & Xia (JFM, 2008)
- Numerical results
  - K41 @ center, BO59 @ boundaries ? Calzavarini et al. (PRE, 2002)
  - Anomalous scalings in HRB: Biferale et al. (EL, 2003)
  - BO59 on temperature, K41 on V: Verzicco & Camussi (JFM, 2003)
- BO59 in both experiments & numerics: Kunnen et al. (PRE, 2008)
- Review by Lohse & Xia (ARFM 2010 in press)
  - The BO59 range, if any, is limited
  - Care is needed to disentangle various effects & measurements biases

This work

- Use existing “tools”
  - SO(3) decomposition of structure functions (e.g. Arad et al., PRE 1999)
  - Quantification of inhomogeneity (Danaila et al., JFM 2001)
  - Scale-by-scale budgets (Casciola et al., JFM 2003)
- Derive generalized Kolmogorov and Yaglom equations
  - Perform SO(3) decomposition of statistical averages
  - Keep the vertical dependence of statistical quantities
- How are these exact laws satisfied in DNS ?
  - Test for K41 or BO59 “tendencies”
  - Quantify simultaneously the effects of anisotropy and inhomogeneity

Theory

- "Classic" derivation in the Boussinesq approximation
  - Write momentum and temperature equations at different x and x′
  - Take the difference & multiply by velocity and temperature increments
  - Average

$$\partial_t \left\langle (\Delta u_i)^2 \right\rangle + \left\langle \partial'_k \left(\Delta u_k (\Delta u_i)^2 \right) \right\rangle + \left\langle (\partial_k + \partial'_k) \left(u_k (\Delta u_i)^2 \right) \right\rangle =$$

$$-\frac{2}{\rho} \left\langle (\partial_i + \partial'_i) (\Delta u_i \Delta p) \right\rangle + 2\alpha g \left\langle \Delta\theta \Delta u_z \right\rangle + \nu \left\langle \left(\partial_k^2 + \partial_k'^2 \right) (\Delta u_i)^2 \right\rangle - 2 \left(\langle \varepsilon \rangle + \langle \varepsilon' \rangle \right)$$

$$\partial_t \left\langle (\Delta\theta)^2 \right\rangle + \left\langle \partial'_k \left(\Delta u_k (\Delta\theta)^2 \right) \right\rangle + \left\langle (\partial_k + \partial'_k) \left(u_k (\Delta\theta)^2 \right) \right\rangle$$

$$+2 \left\langle u'_z \partial'_z \left\langle T' \right\rangle \Delta\theta \right\rangle - 2 \left\langle u_z \partial_z \left\langle T \right\rangle \Delta\theta \right\rangle =$$

$$\kappa \left\langle \left(\partial_k^2 + \partial_k'^2 \right) (\Delta\theta)^2 \right\rangle - 2 \left(\langle N \rangle + \langle N' \rangle \right)$$

(Rincon, J. Fluid Mech. 563 p. 43-69, 2006)

Inhomogeneous contributions

- Following Hill (2002), change variables:

$$\vec{X} = (\vec{x} + \vec{x}')/2$$

$$\vec{r} = \vec{x}' - \vec{x}$$

$$\partial_k = -\frac{\partial}{\partial r_k} + \frac{1}{2}\frac{\partial}{\partial X_k}$$

$$\partial'_k = \frac{\partial}{\partial r_k} + \frac{1}{2}\frac{\partial}{\partial X_k}$$

- Assume Z-dependence of statistical quantities and stationarity

$$\frac{\partial}{\partial r_k}\left\langle \Delta u_k (\Delta u_i)^2 \right\rangle + \frac{1}{2}\frac{\partial}{\partial Z}\left\langle \Delta u_z (\Delta u_i)^2 \right\rangle + \frac{\partial}{\partial Z}\left\langle u_z (\Delta u_i)^2 \right\rangle =$$

$$-2\left(\langle \varepsilon \rangle + \langle \varepsilon' \rangle\right) - \frac{2}{\rho}\frac{\partial}{\partial Z}\langle \Delta u_z \Delta p \rangle + 2\alpha g \langle \Delta u_z \Delta\theta \rangle$$

$$+2\nu \frac{\partial^2}{\partial r_k^2}\left\langle (\Delta u_i)^2 \right\rangle + \frac{\nu}{2}\frac{\partial^2}{\partial Z^2}\left\langle (\Delta u_i)^2 \right\rangle$$

$$\frac{\partial}{\partial r_k}\left\langle \Delta u_k (\Delta\theta)^2 \right\rangle + \frac{1}{2}\frac{\partial}{\partial Z}\left\langle \Delta u_z (\Delta\theta)^2 \right\rangle + \frac{\partial}{\partial Z}\left\langle u_z (\Delta\theta)^2 \right\rangle$$

$$+2\left\langle u'_z \partial'_z \langle T' \rangle \Delta\theta \right\rangle - 2\left\langle u_z \partial_z \langle T \rangle \Delta\theta \right\rangle =$$

$$-2\left(\langle N \rangle + \langle N' \rangle\right) + 2\kappa \frac{\partial}{\partial r_k}\left\langle (\Delta\theta)^2 \right\rangle + \frac{\kappa}{2}\frac{\partial^2}{\partial Z^2}\left\langle (\Delta\theta)^2 \right\rangle$$

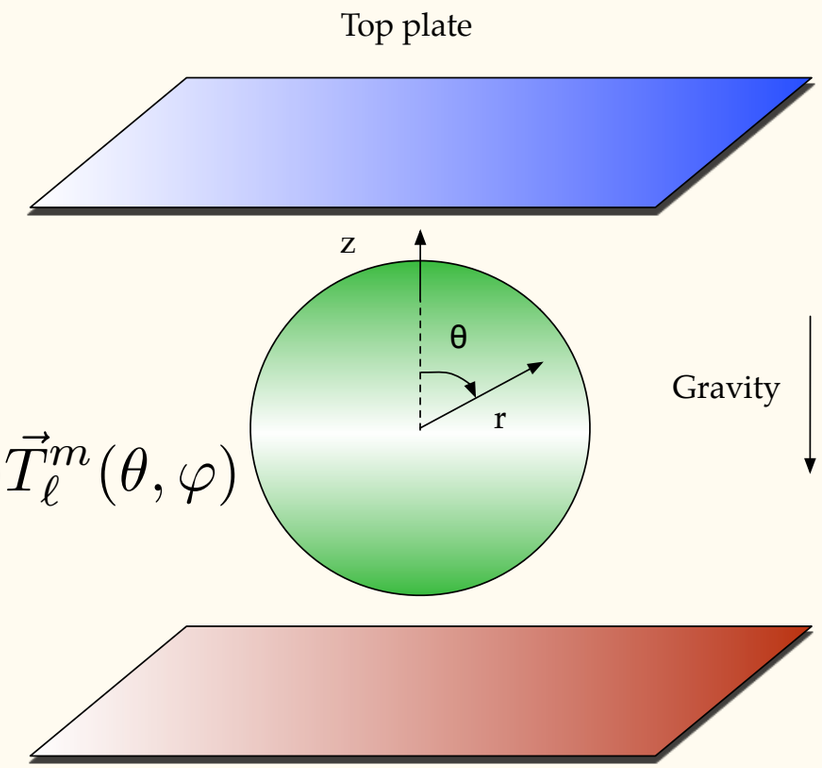
Deal with anisotropy

- Expand every statistical quantity according to

$$\langle F \rangle (Z, \vec{r}) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} \langle F \rangle_m^\ell (Z, r) Y_\ell^m (\theta, \varphi) \qquad \text{Scalars}$$

$$\left\langle \vec{U} \right\rangle (Z, \vec{r}) = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} \langle U_R \rangle_m^\ell (Z, r) \vec{R}_\ell^m (\theta, \varphi) \qquad \text{Vectors}$$

$$+ \langle U_S \rangle_m^\ell (Z, r) \vec{S}_\ell^m (\theta, \varphi) + \langle U_T \rangle_m^\ell (Z, r) \vec{T}_\ell^m (\theta, \varphi)$$



- Useful formulae

$$\left(\frac{\partial \langle U_k \rangle}{\partial r_k} \right)_m^\ell = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \langle U_R \rangle_m^\ell \right) - \frac{\ell(\ell+1)}{r} \langle U_S \rangle_m^\ell \qquad \text{Divergence}$$

$$\left(\frac{\partial^2 \langle F \rangle}{\partial r_k^2} \right)_m^\ell = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \langle F \rangle_m^\ell \right) - \frac{\ell(\ell+1)}{r^2} \langle F \rangle_m^\ell \qquad \text{Laplacian}$$

Final set of equations

$$
\begin{aligned}
\langle U_R\rangle_m^\ell - \frac{\ell(\ell+1)}{r^2}\int_0^r y\,\langle U_S\rangle_m^\ell\,\mathrm{d}y \quad &= \\
&- \frac{2}{r^2}\int_0^r y^2\left(\langle\varepsilon\rangle_m^\ell + \langle\varepsilon'\rangle_m^\ell\right)\mathrm{d}y \\
&+ \frac{2\alpha g}{r^2}\int_0^r y^2\,\langle\Delta u_z\Delta\theta\rangle_m^\ell\,\mathrm{d}y \\
&+ 2\nu\frac{\partial}{\partial r}\left\langle(\Delta u_i)^2\right\rangle_m^\ell - \frac{2\nu\,\ell(\ell+1)}{r^2}\int_0^r\left\langle(\Delta u_i)^2\right\rangle_m^\ell\,\mathrm{d}y \\
&+ \langle NH_\nu\rangle_m^\ell + \langle NH_p\rangle_m^\ell + \langle NH_{u_z}\rangle_m^\ell + \langle NH_{\Delta u_z}\rangle_m^\ell
\end{aligned}
$$

$$
\begin{aligned}
\langle NH_\nu\rangle_m^\ell &= \frac{\nu}{2\,r^2}\int_0^r y^2\frac{\partial^2}{\partial Z^2}\left\langle(\Delta u_i)^2\right\rangle_m^\ell\,\mathrm{d}y\ , \\
\langle NH_p\rangle_m^\ell &= -\frac{2}{\rho\,r^2}\int_0^r y^2\frac{\partial}{\partial Z}\,\langle\Delta u_z\Delta p\rangle_m^\ell\,\mathrm{d}y\ , \\
\langle NH_{u_z}\rangle_m^\ell &= -\frac{1}{r^2}\int_0^r y^2\frac{\partial}{\partial Z}\left\langle u_z(\Delta u_i)^2\right\rangle_m^\ell\,\mathrm{d}y\ , \\
\langle NH_{\Delta u_z}\rangle_m^\ell &= -\frac{1}{2\,r^2}\int_0^r y^2\frac{\partial}{\partial Z}\left\langle\Delta u_z(\Delta u_i)^2\right\rangle_m^\ell\,\mathrm{d}y
\end{aligned}
$$

$$
\left\langle\vec{U}\right\rangle \equiv \left\langle(\Delta u_i)^2\Delta\vec{u}\right\rangle
$$

Final set of equations

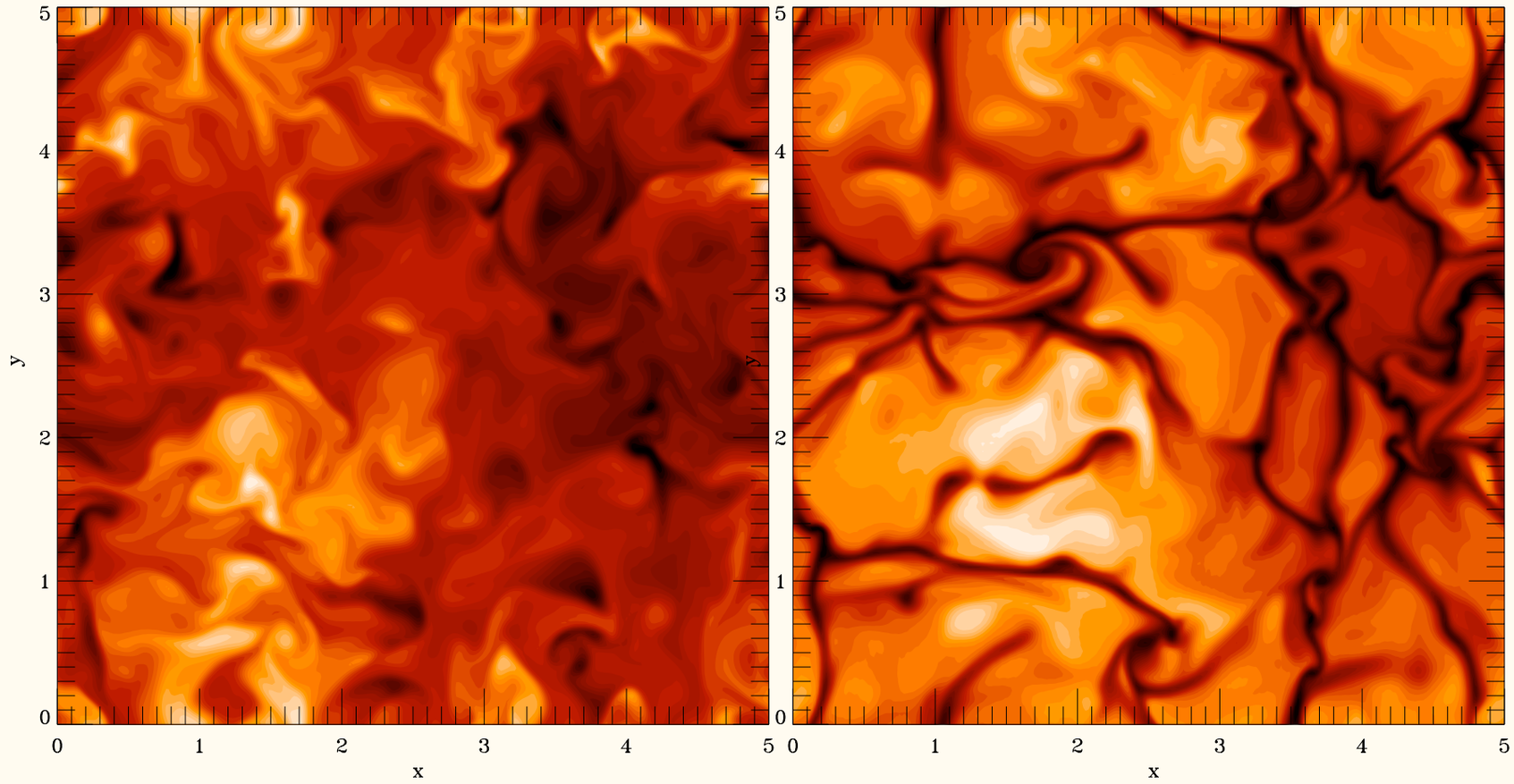
$$
\begin{aligned}
\langle \Theta_R \rangle_m^\ell - \frac{\ell(\ell+1)}{r^2} \int_0^r y \, \langle \Theta_S \rangle_m^\ell \, \mathrm{d}y \quad &= \\
&- \frac{2}{r^2} \int_0^r y^2 \left(\langle N \rangle_m^\ell + \langle N' \rangle_m^\ell \right) \mathrm{d}y \\
&+ \frac{2}{r^2} \int_0^r y^2 \left(\langle u_z \partial_z \langle T \rangle \, \Delta\theta \rangle_m^\ell - \langle u_z' \partial_z' \langle T' \rangle \, \Delta\theta \rangle_m^\ell \right) \mathrm{d}y \\
&+ 2\kappa \frac{\partial}{\partial r} \left\langle (\Delta\theta)^2 \right\rangle_m^\ell - \frac{2\kappa \, \ell(\ell+1)}{r^2} \int_0^r \left\langle (\Delta\theta)^2 \right\rangle_m^\ell \mathrm{d}y \\
&+ \left\langle N H_\kappa^\theta \right\rangle_m^\ell + \left\langle N H_{u_z}^\theta \right\rangle_m^\ell + \left\langle N H_{\Delta u_z}^\theta \right\rangle_m^\ell
\end{aligned}
$$

$$
\begin{aligned}
\left\langle N H_\kappa^\theta \right\rangle_m^\ell &= \frac{\kappa}{2\, r^2} \int_0^r y^2 \frac{\partial^2}{\partial Z^2} \left\langle (\Delta\theta)^2 \right\rangle_m^\ell \mathrm{d}y \ , \\
\left\langle N H_{u_z}^\theta \right\rangle_m^\ell) &= -\frac{1}{r^2} \int_0^r y^2 \frac{\partial}{\partial Z} \left\langle u_z (\Delta\theta)^2 \right\rangle_m^\ell \mathrm{d}y \ , \\
\left\langle N H_{\Delta u_z}^\theta \right\rangle_m^\ell &= -\frac{1}{2\, r^2} \int_0^r y^2 \frac{\partial}{\partial Z} \left\langle \Delta u_z (\Delta\theta)^2 \right\rangle_m^\ell \mathrm{d}y \ .
\end{aligned}
$$

$$
\left\langle \vec{\Theta} \right\rangle \equiv \left\langle (\Delta\theta)^2 \Delta\vec{u} \right\rangle
$$

Numerics - DNS

- Use DNS data to compute all these terms
- Boussinesq 3D run
  - Aspect ratio 5
  - $Ra=10^6$, Pr=1
  - Taylor Re~35
  - 50 turnover times



- "Small" Reynolds numbers but...
  - Every quantity is known at each grid point: exact N-point averages !

Isotropic part - central plane

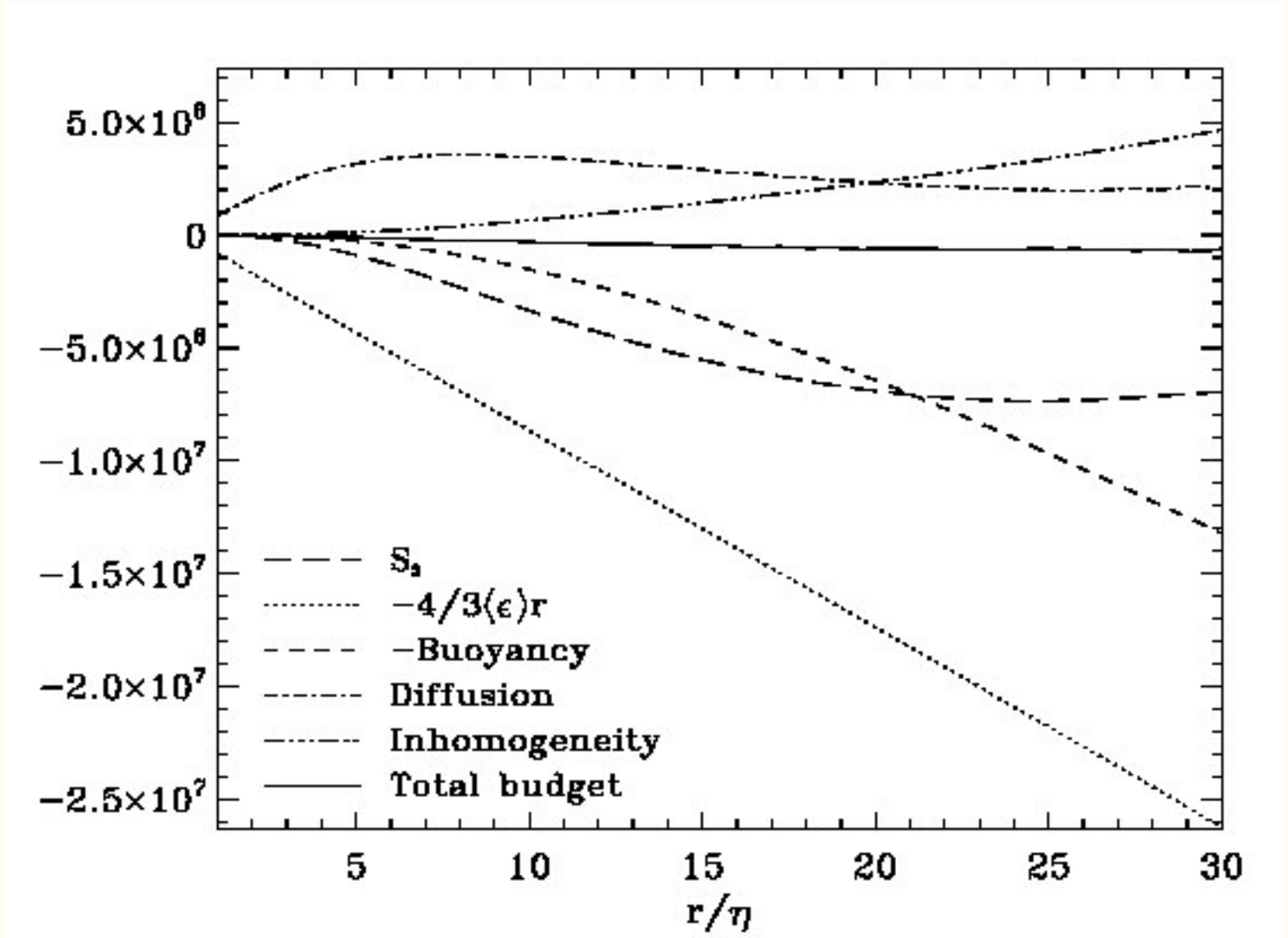
- Spherically-averaged scale-by-scale budget

$$-\langle U_R\rangle_0^0+\frac{2\alpha g}{r^2}\int_0^r y^2\,\langle\Delta u_z\Delta\theta\rangle_0^0\,\mathrm{d}y+2\nu\frac{\partial}{\partial r}\left\langle(\Delta u_i)^2\right\rangle_0^0+\langle NH\rangle_0^0=$$

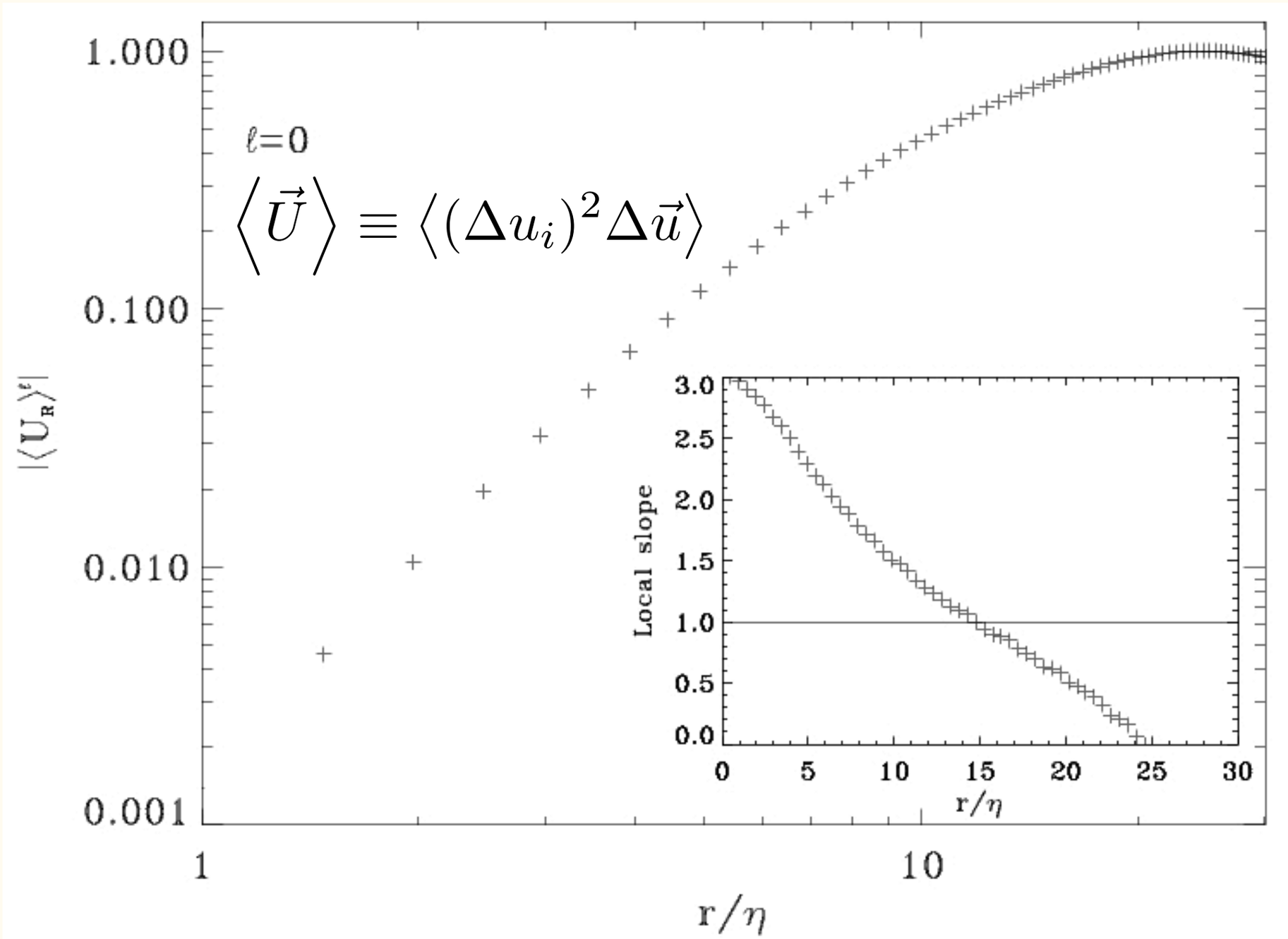
$$\frac{2}{r^2}\int_0^r y^2\left(\langle\varepsilon\rangle_0^0+\langle\varepsilon'\rangle_0^0\right)\mathrm{d}y$$

  - Buoyancy at all scales
  - Inhomogeneity
  - Subdominant viscous term

- $L_B \geq 60\eta$ ~ Box depth
  - No B059 scaling
- Buoyancy interferes
  - No K41 either !



Third-order structure function



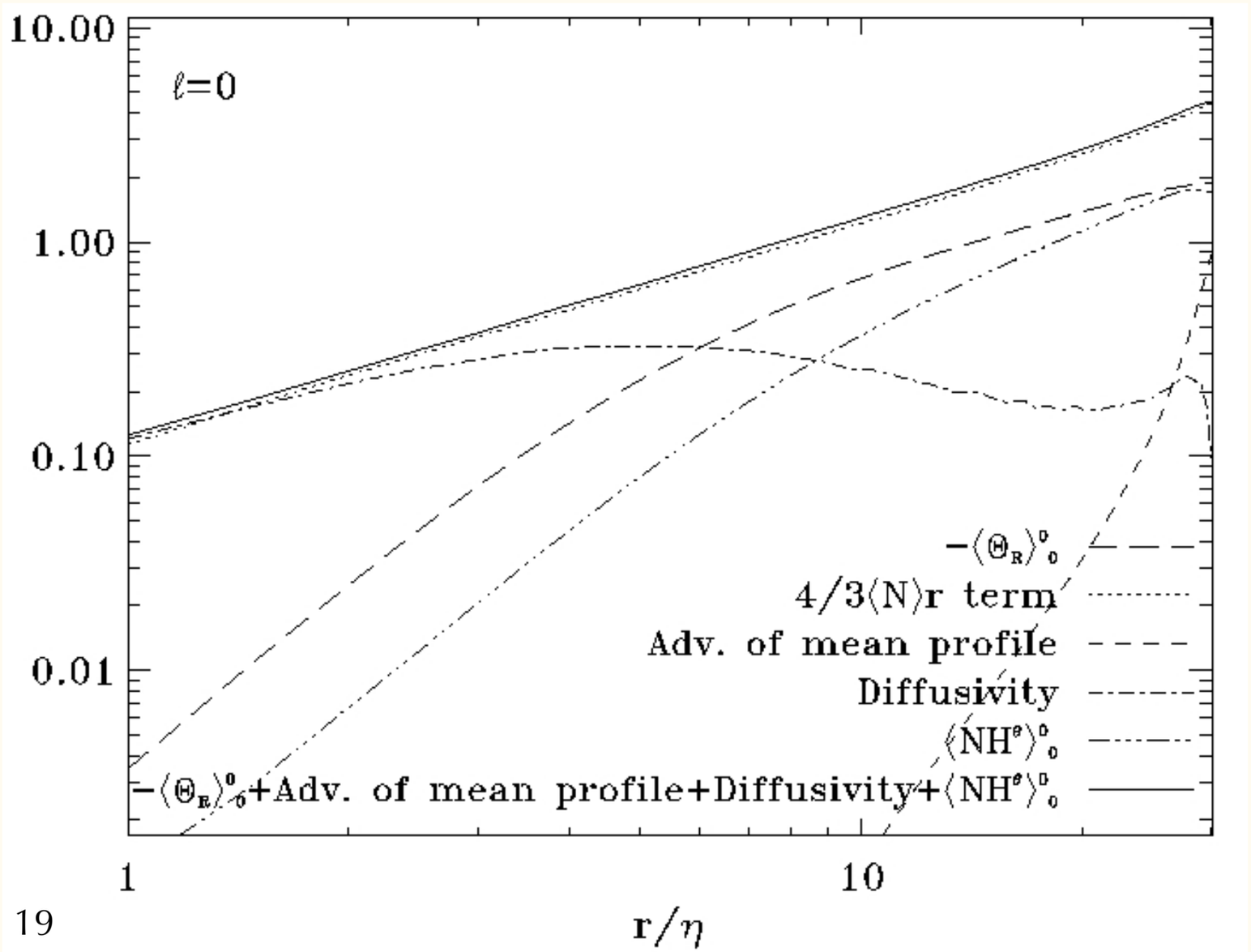
Isotropic part - Temperature

- Spherically-averaged scale-by-scale budget

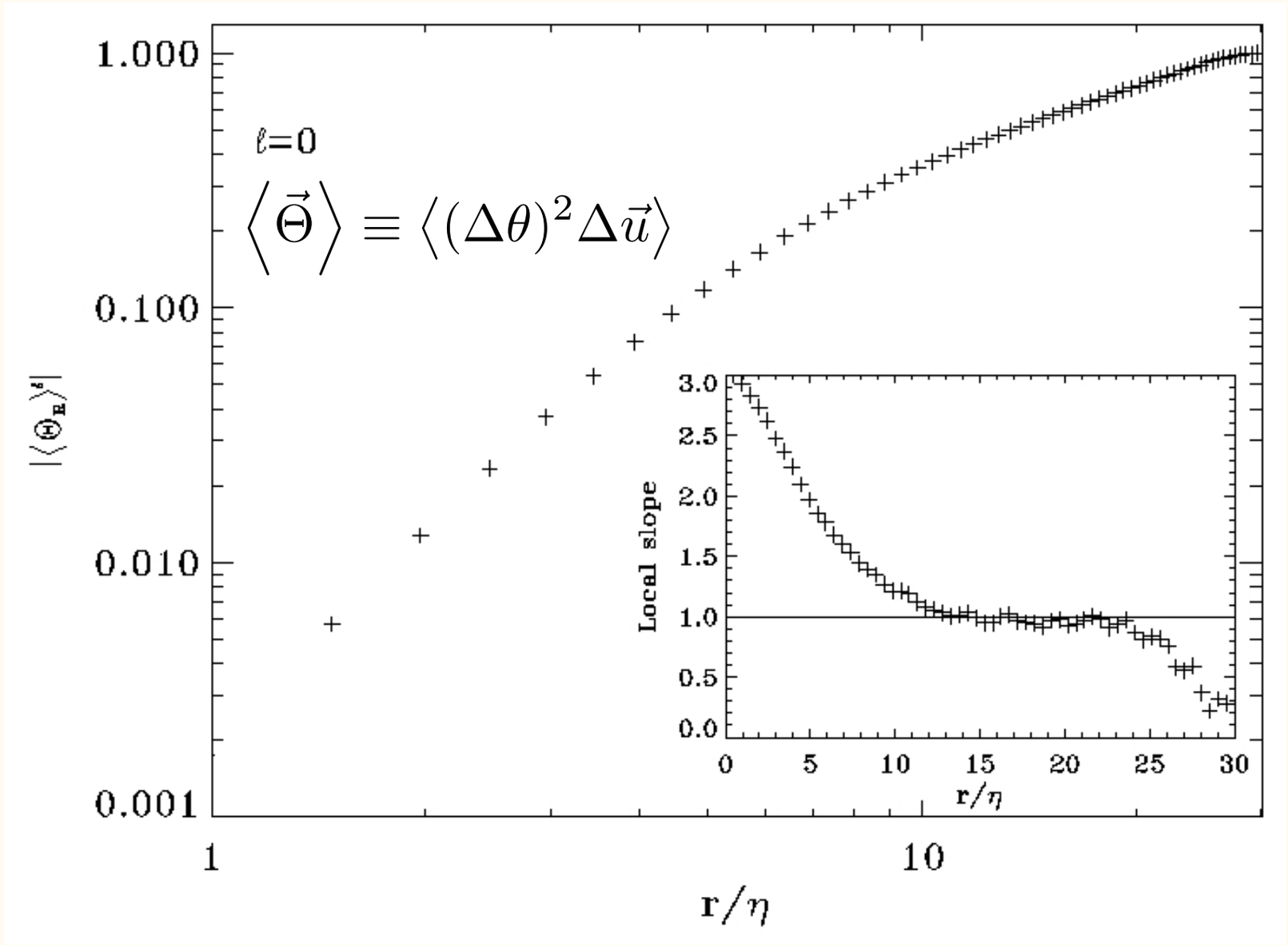
$$-\langle\Theta_R\rangle_0^0+\frac{2}{r^2}\int_0^r y^2\left(\langle u_z\partial_z\langle T\rangle\Delta\theta\rangle_0^0-\langle u_z'\partial_z'\langle T'\rangle\Delta\theta\rangle_0^0\right)\mathrm{d}y$$

$$+2\kappa\frac{\partial}{\partial r}\left\langle(\Delta\theta)^2\right\rangle_0^0+\left\langle NH^{\theta}\right\rangle_0^0=$$

$$\frac{2}{r^2}\int_0^r y^2\left(\langle N\rangle_0^0+\langle N'\rangle_0^0\right)\mathrm{d}y$$



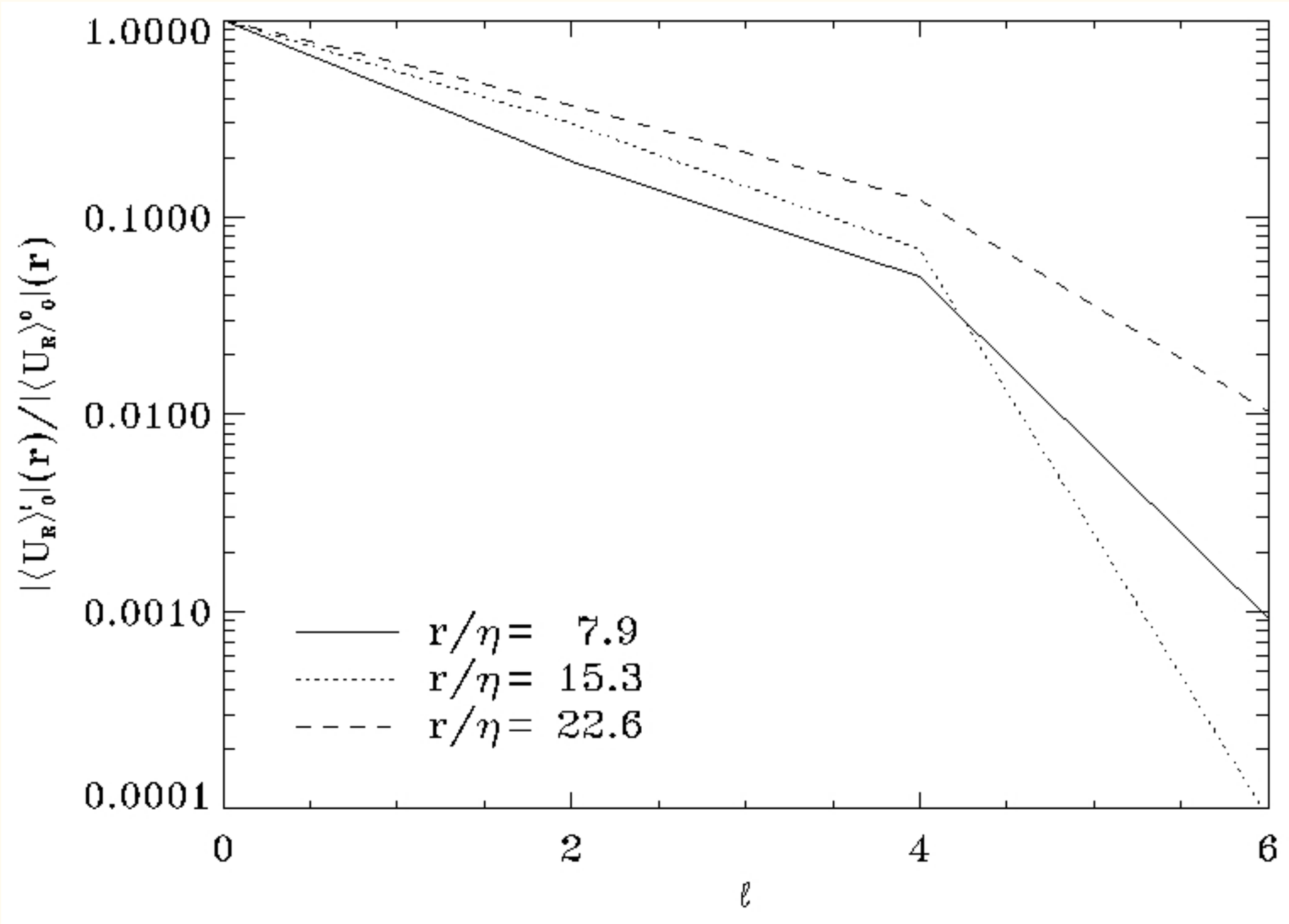
Crossed third-order structure function



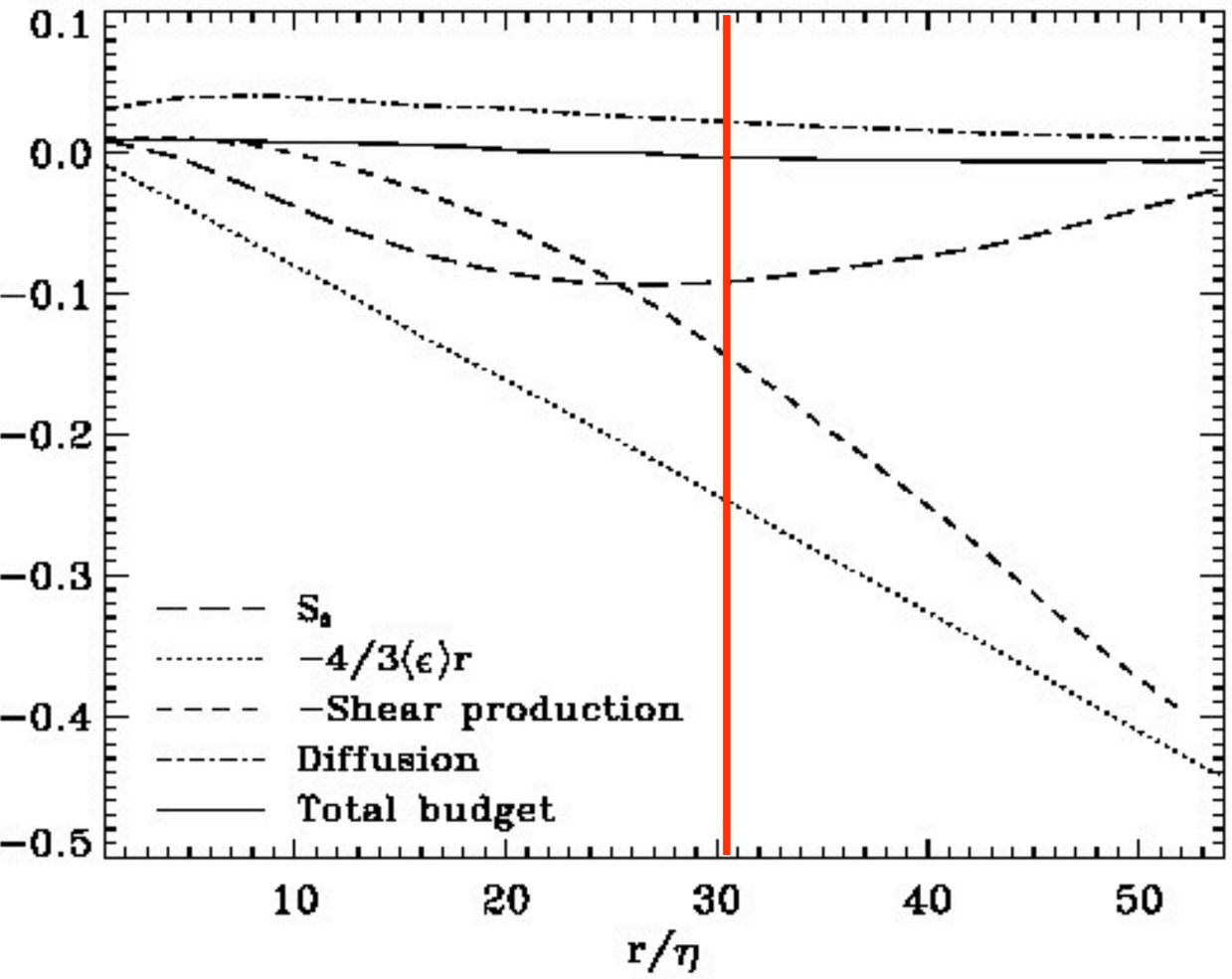
Small scaling range despite diffusion !!

Anisotropy

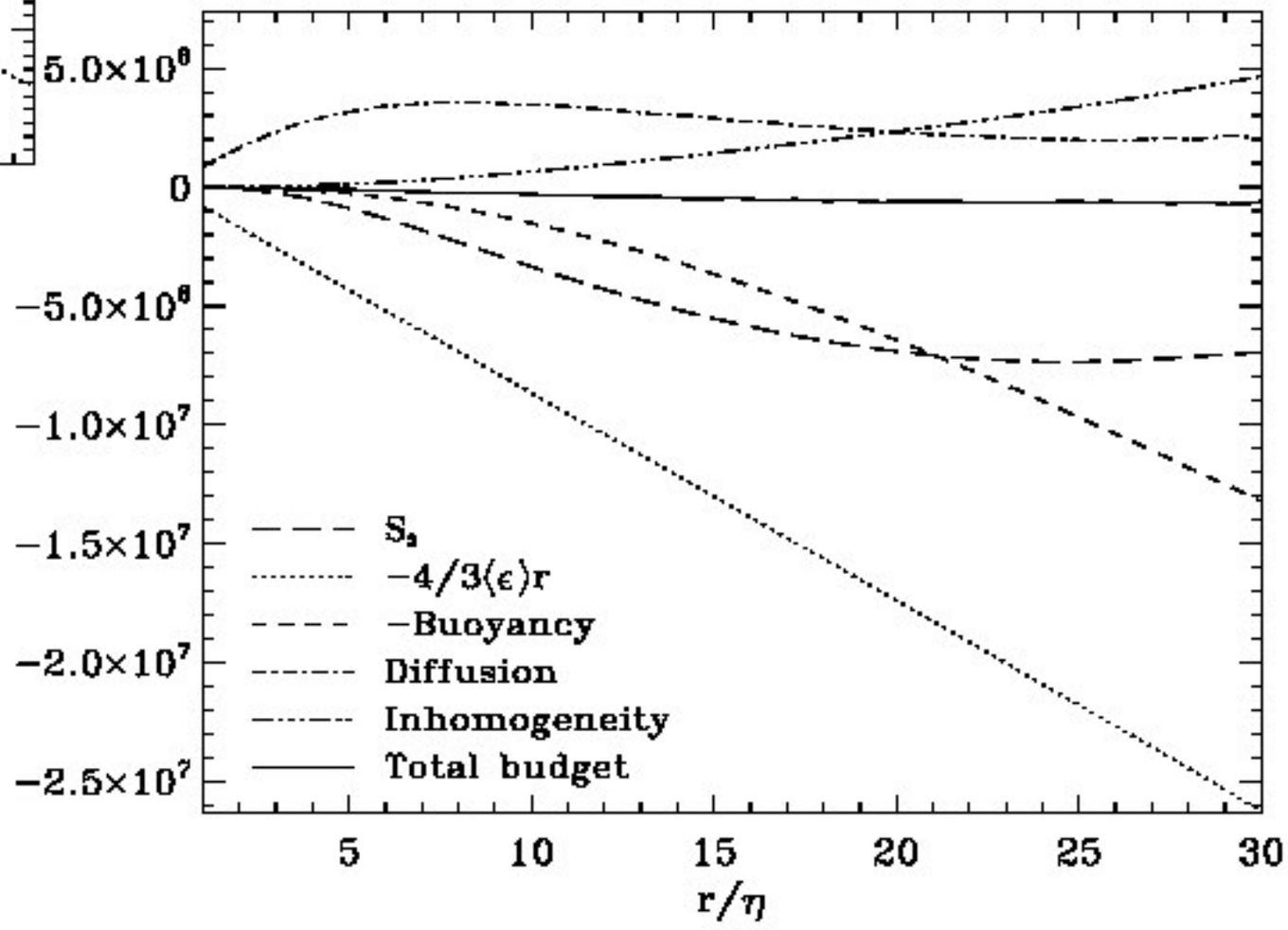
- Normalized amplitudes of different ℓ modes for various r



Homogeneous shear flow turbulence



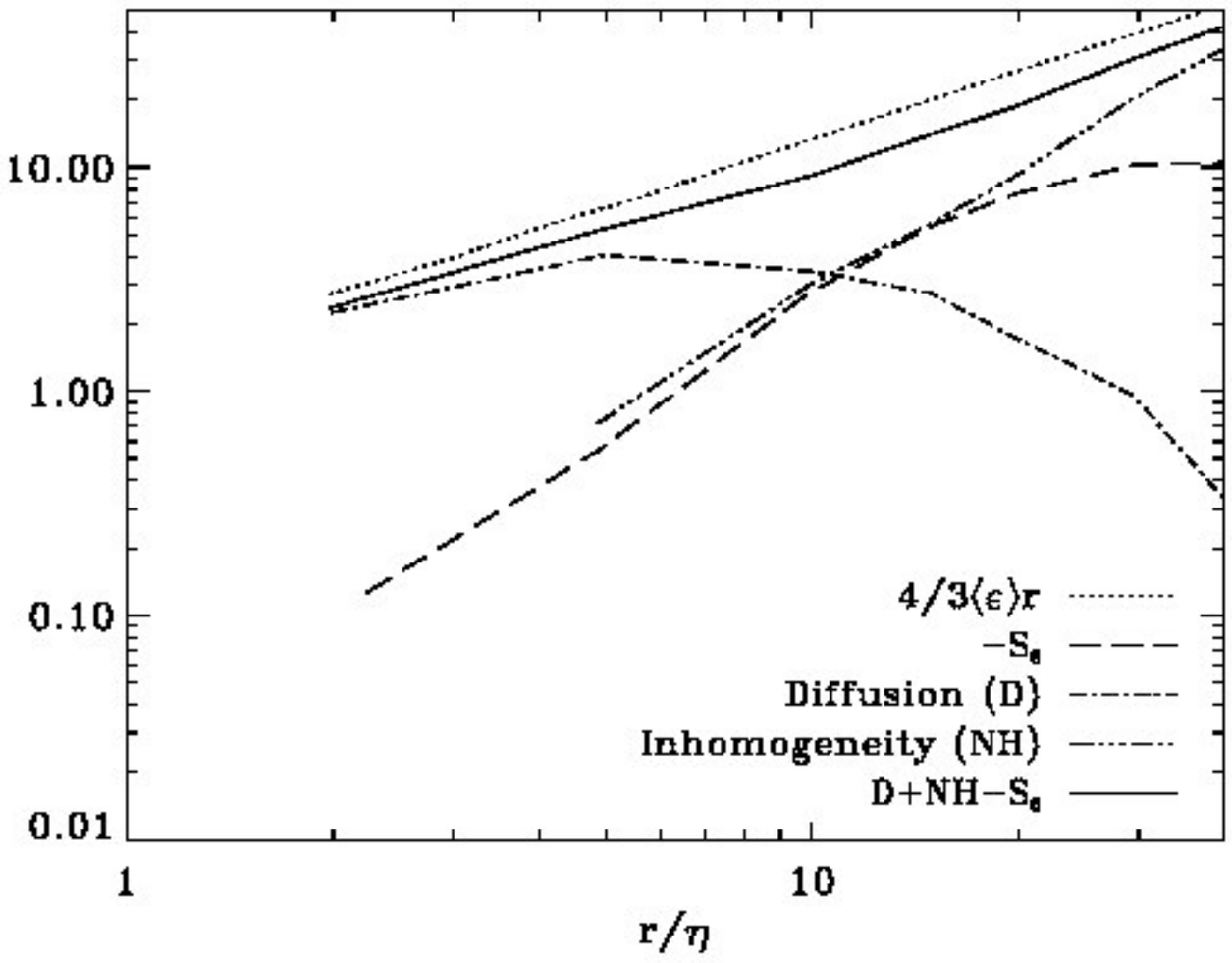
Turbulent spherically-averaged statistics in HSF (from Casciola et al., JFM 2003)



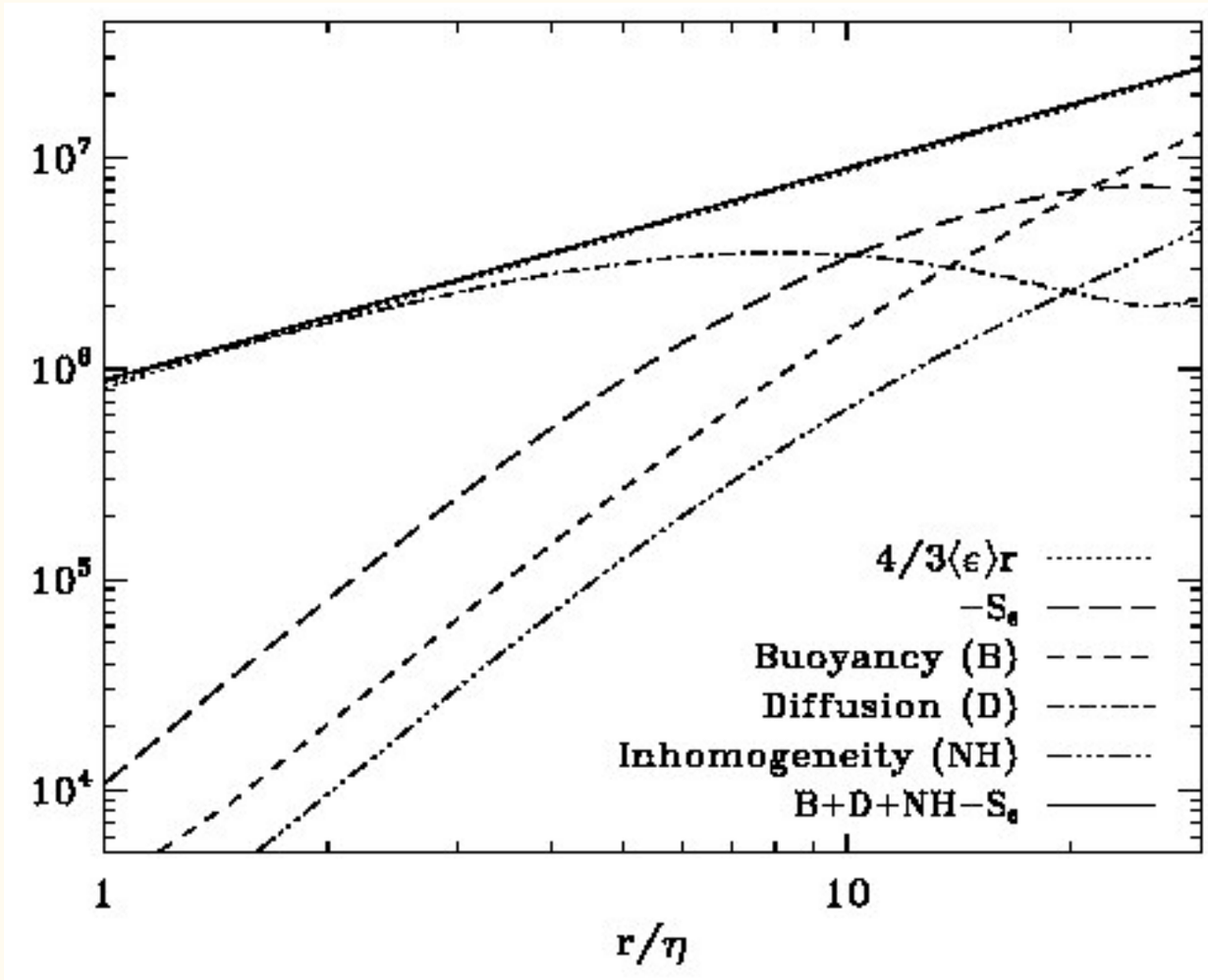
Turbulent spherically-averaged statistics in RBC (Rincon, JFM 2006)

Very similar features !

Channel flow turbulence



Inhomogeneity in channel flow turbulence
(Danaila, JFM 2001)



Turbulent spherically-averaged statistics
in RBC (Rincon, JFM 2006)

Very similar features !

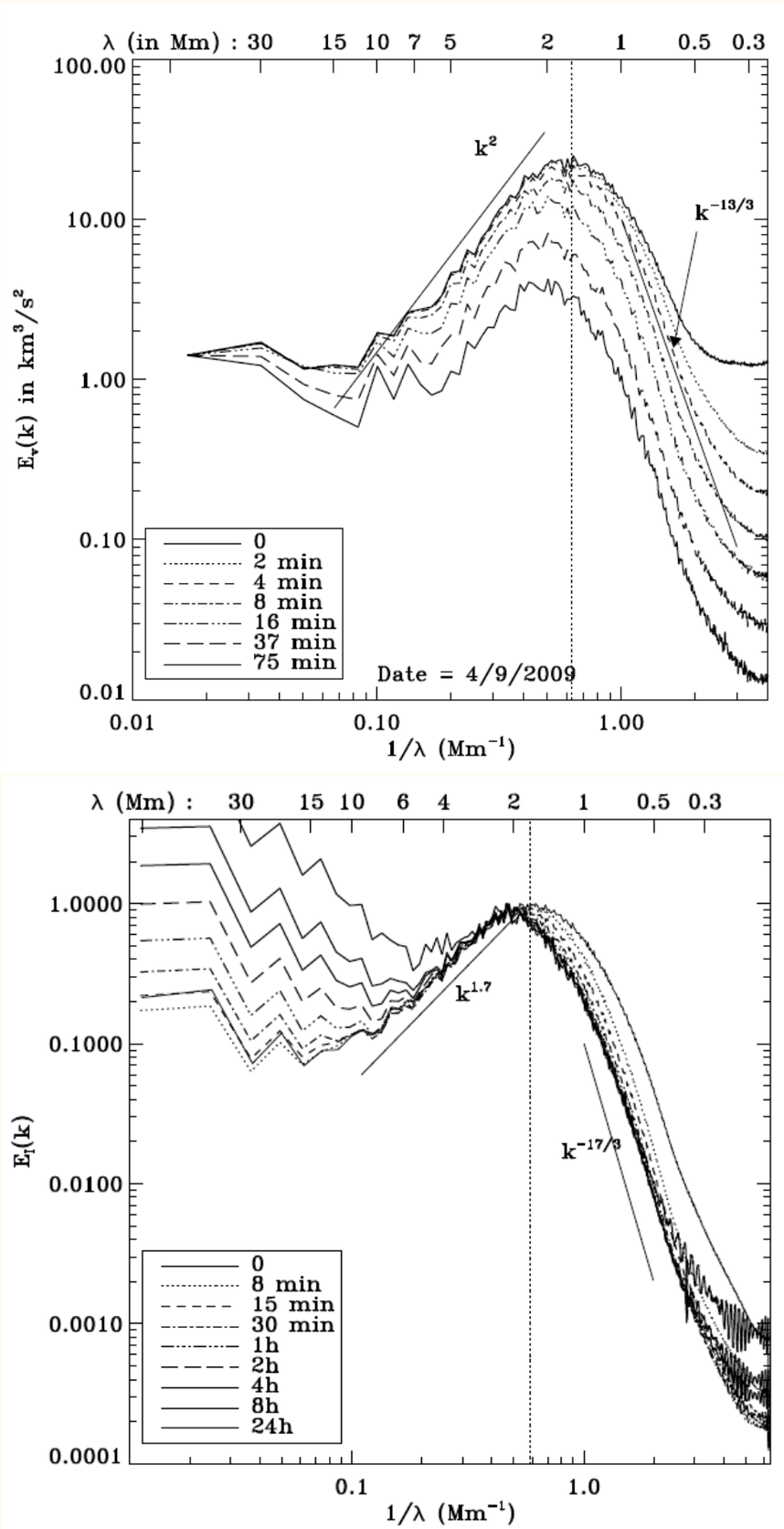
Final word on solar convection

- $\Delta u^3/r \sim \Delta u \Delta\theta$ seems to stand
  - One leg of Bolgiano-Obukhov !
- Dominant balance in Yaglom equation ?
  - Clearly, not the BO59 $\langle \Delta\theta^2 \Delta u \rangle \sim -4/3 \langle N \rangle r$
  - Pr << 1: negligible temperature nonlinearity ?

$$\vec{v}_z \cdot \vec{\nabla}_z \overline{T} \sim \kappa \nabla^2 \theta + \vec{\nabla}\kappa \cdot \vec{\nabla}\theta$$

  - Strong density stratification & Non-adiabaticity
    - Background varies on same scales as fluctuations

Rieutord, Roudier, Rincon et al., A&A 2009 accepted



The end