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Relating statistics to dynamics in axisymmetric homogeneous turbulence

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Abstract. *Homogeneous turbulence submitted to distorsions such as solid body rotation, stratification, or the Lorentz force in the MHD context, exhibit axisymmetric statistics, as a clear departure from isotropy. These anisotropic effects that arise due to a modified dynamics and energy exchange do not fall within the classical description of turbulence in Kolmogorov's theory. We suggest to extend the description of the velocity increment statistics to the anisotropic case, and to relate them to the modified dynamics in axisymmetric homogeneous turbulence, especially energy and transfer spectra.* Keywords*:* turbulence, anisotropy.

In homogeneous initially isotropic turbulence, isotropy can be broken by introducing an external distorsion on the flow: solid body rotation is present for instance in geophysical flows and acts through the Coriolis force, as well as a buoyancy force in density- or temperature- stratified layers. In conducting fluids, the action of an external magnetic field also modifies the symmetries by means of the Lorentz force. In these three examples, the intensity of the corresponding force depends on the orientation of the fluid motion with respect to either the rotation axis, the gravity axis, the background magnetic field axis. Wave propagation can also be present, *e.g.* inertio-gravity waves or Alfvén waves. They provide an anisotropic way of redistributing energy in terms of scale and direction, such that the dynamics of energy exchange in the turbulence is strongly modified.

Taking into account these modifications can be done at different levels. The statistical description of the velocity field distribution can be done at a two-point level in physical space by the second-order velocity correlation tensor $R_{ij}(\boldsymbol{r}) = \langle u_i(\boldsymbol{x})u_j(\boldsymbol{x}+\boldsymbol{r}\rangle$ where $\boldsymbol{r}$ is the separation vector. Fourier-transforming this tensor yields the spectrum tensor $\Phi_{ij}(\boldsymbol{k} = 1/(8\pi)^3 \int R_{ij}(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\mathrm{d}\boldsymbol{r}$, whose trace is $\Phi_{ii}(\boldsymbol{k}) = E(k)/2\pi k^2$ in isotropic turbulence, with $k = |\boldsymbol{k}|$ and $E(k)$ is the kinetic energy spectrum. Kolmogorov theory supports the scaling $E(k) \sim k^{-5/3}$ in isotropic turbulence at high Reynolds number. However, when dealing with statistically axisymmetric turbulence, the orientation θ of the Fourier vector $\boldsymbol{k}$ with respect to the axis of symmetry borne by, say, $\boldsymbol{n}$, has

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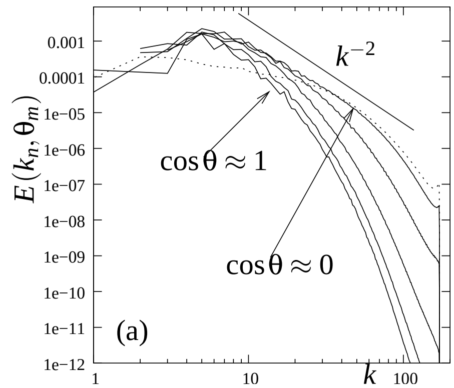

Figure 1: Anisotropic energy spectrum $E(k,\theta)$ in rotating turbulence at small Rossby number. Data from 256^3 DNS. (k_n and θ_n refer to discretized Fourier space.)

to be taken into account, so that an additional dependence has to be introduced in the energy spectrum $E(k,\theta)$. Integrating the latter over the θ domain of course provides an equivalent $E(k)$. The question of the resulting scaling of the spectrum is raised. Figure 1 shows an example of an anisotropic spectrum in rotating turbulence in which several scalings are observed for $E(k,\theta)$ depending on θ, ranging from k^{-2} for horizontal wavevectors to k^{-5} for vertical ones [1]. Note the strong anisotropy down to the smallest scales, which can not only be explained by a low Reynolds number artefact, but also by a differential nonlinear effect of rotation on large and small structures. Nonlinear energy transfer $T(k)$ appears in the Lin equation as

$$\partial_t E(k) + 2\nu k^2 E(k) = T(k) \tag{1}$$

in the isotropic case, where ν is the viscosity, such that the total dissipation is $\varepsilon = 2\nu \int k^2 E(k) \mathrm{d}k$. In the rotating case for instance, phase-scrambling anisotropically changes the shape of the energy exchange term $T(k,\theta)$ in an equation equivalent to (1). The transfer contains third-order correlation terms, and is thus related to the third order statistics that are used to quantify nonlinearity such as the skewness of velocity gradient distribution.

Equation (1) therefore relates the evolution of second-order statistics—the energy spectrum—to third-order ones—the transfer. The analogous in physical space of (1) is therefore the Karman-Howarth equation

$$\partial_t \left(u'^2 f\right) = \left(\partial_r + \frac{4}{r}\right) \left[R_{LL,L}(r,t) + 2\nu \partial_r \left(u'^2 f\right)\right] \tag{2}$$

in which f is the longitudinal two-point correlation function; the longitudinal two-point third-order correlation function is $R_{LL,L}(r,t) = \langle u_i(\boldsymbol{x},t)u_m(\boldsymbol{x},t)u_i(\boldsymbol{x}+\boldsymbol{r},t)\rangle r_m/r$, $r = |\boldsymbol{r}|$, and $(3/2)u'^2$ is the total kinetic energy. Upon examining (1) and (2) together, it is clear that if some directional dependence with $\boldsymbol{k}$ appears in the former for $E(\boldsymbol{k})$, then anisotropy with respect to $\boldsymbol{r}$ is also expected in the latter.

The dependence of the velocity, or velocity increments $\delta\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$, on the direction of $\boldsymbol{r}$ with respect to $\boldsymbol{n}$, is therefore an important parameter in axisymmetric turbulence statistics. Kolmogorov's theory (1941) only considers a scalar isotropic longitudinal increment $\delta u_\parallel$, provides the scaling $\langle(\delta u_\parallel)^n\rangle \sim (\varepsilon r)^{n/3}$, and allows to draw from equation (2) a simplified relationship between second- and third-order structure functions

$$\langle\left(\delta u_\parallel\right)^3\rangle = -\frac{4}{5}\varepsilon r + 6\nu\partial_r\langle(\delta u_\parallel)^2\rangle.$$

which yields, at infinite Reynolds number, the famous four fifths law $\langle\left(\delta u_\parallel\right)^3\rangle = -(4/5)\varepsilon r$, or $\langle\delta u_\parallel\,(\delta q)^2\rangle = -(4/3)\varepsilon r$ with $\delta q = (\delta u_i)(\delta u_i)$. [2]

A useful relationship can be established between third-order spectral statistics *i.e.* the nonlinear energy transfer $T(k)$, and the third-order structure function :

$$\langle\delta u_\parallel\,(\delta q)^2\rangle = 4r\int_0^\infty g(kr)T(k)\mathrm{d}k \tag{3}$$

where $g(kr) = (\sin kr - kr\cos kr)/(kr)^3$. It shows the direct link of the third-order structure function on the dynamics of turbulence. In anisotropic turbulence, the analogous of (3) implies a vector dependence of the structure functions on $\boldsymbol{r}$. It also shows that a dynamics modified by an external distorsion applied to homogeneous turbulence translates immediatly in different scalings of the structure functions. It is therefore interesting to discuss the applicability of Kolmogorov scalings in flows that contain some anisotropy. In the axisymmetric case, an extensive work is in progress [3].

References

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